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Bi-local Fields in Noncommutative Field Theory

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Abstract

We propose a bi-local representation in noncommutative field theory. It provides a simple description for high momentum degrees of freedom. It also shows that the low momentum modes can be well approximated by ordinary local fields. Long range interactions are generated in the effective action for the lower momentum modes after integrating out the high momentum bi-local fields. The low momentum modes can be represented by diagonal blocks in the matrix model picture and the high momentum bi-local fields correspond to off-diagonal blocks. This block-block interaction picture simply reproduces the infrared singular behaviors of nonplanar diagrams in noncommutative field theory.

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1 Introduction

Superstring theory is a leading candidate for the unification of fundamental interactions. As such it is expected to reconcile quantum mechanics and general relativity. As is well known, field theoretic approach to quantize gravity encounters serious difficulties at short distances. String theory contains the notion of the minimal length (string scale) which is expected to cure such difficulties.

Although the first quantization of string theory is well understood, it is ripe to obtain fully nonperturbative formulation of superstring theory. We have proposed such a formalism as a large N reduced model with the maximal chiral SUSY [1]. It is a finite theory which has a potential to predict the structure of space-time[2][3].

Another school of thought which also introduces the minimum length scale advocates to replace Riemannian geometry by noncommutative geometry[4]. Furthermore string theory in noncommutative tori has been studied in [5].

Remarkably, these various schools of thought have converged recently. We have pointed out that noncommutative Yang-Mills theory is naturally obtained in IIB matrix model by expanding the theory around a noncommutative space-time[6]. We have further studied the Wilson loops. These investigations have shown that the theory contains not only point like field theoretic degrees of freedom but also open string like extended objects [7]. In terms of the matrix representation, the former is close to diagonal and the latter is far off-diagonal degrees of freedom. In string theory these backgrounds are interpreted as D-branes with constant $b_{\mu\nu}$ field background[8]. These problems are further studied in [9][10].

In this paper, we continue our investigation of noncommutative field theory as twisted reduced models. We have mapped the twisted reduced model onto the noncommutative field theory by expanding the matrices by the momentum eigenstates. In this paper we propose a more natural decomposition rule for the matrices in terms of a bi-local basis. We show that they corresponds to 'open strings' and we can reproduce quenched reduced models with long 'open strings'. We also show that the diagonal elements represent ordinary plane waves. Noncommutative field theory exhibits crossover at the noncommutative scale. At large momentum scale, it is identical to large N field theory and the relevant degrees of freedom are described by 'open strings' longer than the noncommutative scale while it reduces to field theory in the opposite limit. We make these statements more precise in this paper. The equivalence to large N field theory at large momentum scale is further supported

since we find quenched reduced models in noncommutative field theory. In the low energy limit, we find long range interactions in noncommutative field theory which are absent in the ordinary field theory. We point out that this effect can be simply understood in terms of block-block interactions in the matrix model picture.

2 Noncommutative field theories as twisted reduced models

In this section, we briefly recapitulate our approach to noncommutative field theory as twisted reduced models. Reduced models are defined by the dimensional reduction of d dimensional gauge theory down to zero dimension (a point)[11]. The application of reduced models to string theory is pioneered in [12][13]. We consider d dimensional $U(n)$ gauge theory coupled to adjoint matter as an example:

$$S = -\int d^d x \frac{1}{g^2} Tr(\frac{1}{4}[D_\mu, D_\nu][D_\mu, D_\nu] + \frac{1}{2}\bar{\psi}\Gamma_\mu [D_\mu, \psi]), \tag{2.1}$$

where ψ is a Majorana spinor field. The corresponding reduced model is

$$S = -\frac{1}{g^2} Tr(\frac{1}{4}[A_\mu, A_\nu][A_\mu, A_\nu] + \frac{1}{2}\bar{\psi}\Gamma_\mu [A_\mu, \psi]). \tag{2.2}$$

Now A_μ and ψ are $n \times n$ Hermitian matrices and each component of ψ is d-dimensional Majorana-spinor. IIB matrix model is obtained when $d = 10$ and ψ is further assumed to be Weyl-spinor as well. Any noncommutative field theory is realized as well in terms of a twisted reduced model by the same mapping rules as below but here we explain the case of the above reduced models of gauge theory. We expand the theory around the following classical solution:

$$[\hat{p}_\mu, \hat{p}_\nu] = iB_{\mu\nu}, \tag{2.3}$$

where $B_{\mu\nu}$ are c-numbers. We assume the rank of $B_{\mu\nu}$ to be $\tilde{d}$ and define its inverse $C^{\mu\nu}$ in $\tilde{d}$ dimensional subspace. The directions orthogonal to the subspace is called the transverse directions. $\hat{p}_\mu$ satisfy the canonical commutation relations and they span the $\tilde{d}$ dimensional phase space. The semiclassical correspondence shows that the volume of the phase space is $V_p = n(2\pi)^{\tilde{d}/2}\sqrt{detB}$.

We expand $A_\mu = \hat{p}_\mu + \hat{a}_\mu$. We Fourier decompose $\hat{a}_\mu$ and $\hat{\psi}$ fields as

$$\begin{aligned} \hat{a} &= \sum_k \tilde{a}(k) exp(iC^{\mu\nu} k_\mu \hat{p}_\nu), \\ \hat{\psi} &= \sum_k \tilde{\psi}(k) exp(iC^{\mu\nu} k_\mu \hat{p}_\nu), \end{aligned} \tag{2.4}$$

where $exp(iC^{\mu\nu}k_\mu\hat{p}_\nu)$ is the eigenstate of adjoint $P_\mu = [\hat{p}_\mu, \ ]$ with the eigenvalue k_μ. The Hermiticity requires that $\tilde{a}^*(k) = \tilde{a}(-k)$ and $\tilde{\psi}^*(k) = \tilde{\psi}(-k)$. Let us consider the case that $\hat{p}_\mu$ consist of $\tilde{d}/2$ canonical pairs $\hat{p}_i, \hat{q}_i$ which satisfy $[\hat{p}_i, \hat{q}_j] = iB\delta_{ij}$. We also assume that the solutions possess the discrete symmetry which exchanges canonical pairs and $\hat{p}_i \leftrightarrow \hat{q}_i$ in each canonical pair. We then find $V_p = \Lambda^{\tilde{d}}$ where Λ is the extension of each $\hat{p}_\mu$. The volume of the unit quantum in phase space is $\Lambda^{\tilde{d}}/n = \lambda^{\tilde{d}}$ where λ is the spacing of the quanta. B is related to λ as $B = \lambda^2/(2\pi)$. Let us assume the topology of the world sheet to be $T^{\tilde{d}}$ in order to determine the distributions of k_μ. Then we can formally construct $\hat{p}_\mu$ through unitary matrices as $\gamma_\mu = exp(i2\pi\hat{p}_\mu/\Lambda)$. The polynomials of γ_μ are the basis of $exp(iC^{\mu\nu}k_\mu\hat{p}_\nu)$. We can therefore assume that k_μ is quantized in the unit of $|k^{min}| = \lambda/n^{1/\tilde{d}}$. The eigenvalues of $\hat{p}_\mu$ are quantized in the unit of $\Lambda/n^{2/\tilde{d}} = \lambda/n^{1/\tilde{d}}$. Hence we restrict the range of k_μ as $-n^{1/\tilde{d}}\lambda/2 < k_\mu < n^{1/\tilde{d}}\lambda/2$. So $\sum_k$ runs over n^2 degrees of freedom which coincide with those of n dimensional Hermitian matrices.

We can construct a map from a matrix to a function as

$$\hat{a} \rightarrow a(x) = \sum_k \tilde{a}(k) exp(ik\cdot x), \tag{2.5}$$

where $k\cdot x = k_\mu x^\mu$. By this construction, we obtain the $\star$ product

$$\begin{aligned} \hat{a}\hat{b} &\rightarrow a(x)\star b(x), \\ a(x)\star b(x) &\equiv exp(\frac{C^{\mu\nu}}{2i}\frac{\partial^2}{\partial\xi^\mu\partial\eta^\nu})a(x+\xi)b(x+\eta)|_{\xi=\eta=0}. \end{aligned} \tag{2.6}$$

The operation Tr over matrices can be exactly mapped onto the integration over functions as

$$Tr[\hat{a}] = \sqrt{detB}(\frac{1}{2\pi})^{\frac{\tilde{d}}{2}} \int d^{\tilde{d}}x a(x). \tag{2.7}$$

The twisted reduced model can be shown to be equivalent to noncommutative Yang-Mills by the the following map from matrices onto functions

$$\begin{aligned} \hat{a} &\rightarrow a(x), \\ \hat{a}\hat{b} &\rightarrow a(x)\star b(x), \\ Tr &\rightarrow \sqrt{detB}(\frac{1}{2\pi})^{\frac{\tilde{d}}{2}} \int d^{\tilde{d}}x. \end{aligned} \tag{2.8}$$

The following commutator is mapped to the covariant derivative:

$$[\hat{p}_\mu + \hat{a}_\mu, \hat{o}] \rightarrow \frac{1}{i}\partial_\mu o(x) + a_\mu(x)\star o(x) - o(x)\star a_\mu(x) \equiv [D_\mu, o(x)], \tag{2.9}$$

We may interpret the newly emerged coordinate space as the semiclassical limit of $\hat{x}^\mu = C^{\mu\nu}\hat{p}_\nu$. The space-time translation is realized by the following unitary operator:

$$exp(i\hat{p}\cdot d)\hat{x}^\mu exp(-i\hat{p}\cdot d) = \hat{x}^\mu + d^\mu. \tag{2.10}$$

Applying the rule eq.(2.8), the bosonic action becomes

$$\begin{aligned} & -\frac{1}{4g^2}Tr[A_\mu, A_\nu][A_\mu, A_\nu] \\ = \; & \frac{\tilde{d}nB^2}{4g^2} - \sqrt{detB}(\frac{1}{2\pi})^{\frac{\tilde{d}}{2}}\int d^{\tilde{d}}x\frac{1}{g^2}(\frac{1}{4}[D_\alpha, D_\beta][D_\alpha, D_\beta] \\ & +\frac{1}{2}[D_\alpha, \varphi_\nu][D_\alpha, \varphi_\nu] + \frac{1}{4}[\varphi_\nu, \varphi_\rho][\varphi_\nu, \varphi_\rho])_\star. \end{aligned} \tag{2.11}$$

In this expression, the indices α, β run over $\tilde{d}$ dimensional world volume directions and ν, ρ over the transverse directions. We have replaced $a_\nu \to \varphi_\nu$ in the transverse directions. Inside $(\;)_\star$, the products should be understood as $\star$ products and hence commutators do not vanish. The fermionic action becomes

$$\begin{aligned} & \frac{1}{g^2}Tr\bar{\psi}\Gamma_\mu[A_\mu, \psi] \\ = \; & \sqrt{detB}(\frac{1}{2\pi})^{\frac{\tilde{d}}{2}}\int d^{\tilde{d}}x\frac{1}{g^2}(\bar{\psi}\Gamma_\alpha[D_\alpha, \psi] + \bar{\psi}\Gamma_\nu[\varphi_\nu, \psi])_\star. \end{aligned} \tag{2.12}$$

We therefore find noncommutative U(1) gauge theory.

In order to obtain noncommutative Yang-Mills theory with $U(m)$ gauge group, we consider new classical solutions which are obtained by replacing each element of $\hat{p}_\mu$ by the $m\times m$ unit matrix:

$$\hat{p}_\mu \to \hat{p}_\mu \otimes \mathbf{1}_m. \tag{2.13}$$

We require $N = mn$ dimensional matrices for this construction. The fluctuations around this background $\hat{a}$ and $\hat{\psi}$ can be Fourier decomposed in the analogous way as in eq.(2.4) with m dimensional matrices $\tilde{a}(k)$ and $\tilde{\psi}(k)$ which satisfy $\tilde{a}(-k) = \tilde{a}^\dagger(k)$ and $\tilde{\psi}(-k) = \tilde{\psi}^\dagger(k)$. It is then clear that $[\hat{p}_\mu + \hat{a}_\mu, \hat{o}]$ can be mapped onto the nonabelian covariant derivative $[D_\mu, o(x)]$ once we use $\star$ product. Applying our rule (2.8) to the action in this case, we obtain

$$\begin{aligned} & \frac{\tilde{d}NB^2}{4g^2} - \sqrt{detB}(\frac{1}{2\pi})^{\frac{\tilde{d}}{2}}\int d^{\tilde{d}}x\frac{1}{g^2}tr(\frac{1}{4}[D_\alpha, D_\beta][D_\alpha, D_\beta] \\ & +\frac{1}{2}[D_\alpha, \varphi_\nu][D_\alpha, \varphi_\nu] + \frac{1}{4}[\varphi_\nu, \varphi_\rho][\varphi_\nu, \varphi_\rho] \\ & +\frac{1}{2}\bar{\psi}\Gamma_\alpha[D_\alpha, \psi] + \frac{1}{2}\bar{\psi}\Gamma_\nu[\varphi_\nu, \psi])_\star. \end{aligned} \tag{2.14}$$

where tr denotes taking trace over m dimensional subspace. The Yang-Mills coupling is found to be $g_{NC}^2 = (2\pi)^{\frac{\tilde{d}}{2}} g^2/B^{\tilde{d}/2}$. Therefore it will decrease if the density of quanta in phase space decreases with fixed g^2.

The Hermitian models are invariant under the unitary transformation: $A_\mu \to U A_\mu U^\dagger, \psi \to U\psi U^\dagger$. As we shall see, the gauge symmetry can be embedded in the $U(N)$ symmetry. We expand $U = exp(i\hat{\lambda})$ and parameterize

$$\hat{\lambda} = \sum_k \tilde{\lambda}(k) exp(ik \cdot \hat{x}). \tag{2.15}$$

Under the infinitesimal gauge transformation, we find the fluctuations around the fixed background transform as

$$\begin{aligned} \hat{a}_\mu &\to \hat{a}_\mu + i[\hat{p}_\mu, \hat{\lambda}] - i[\hat{a}_\mu, \hat{\lambda}], \\ \hat{\psi} &\to \hat{\psi} - i[\hat{\psi}, \hat{\lambda}]. \end{aligned} \tag{2.16}$$

We can map these transformations onto the gauge transformation in noncommutative Yang-Mills by our rule eq.(2.8):

$$\begin{aligned} & a_\alpha(x) \to a_\alpha(x) + \frac{\partial}{\partial x^\alpha}\lambda(x) - i a_\alpha(x) \star \lambda(x) + i\lambda(x) \star a_\alpha(x), \\ & \varphi_\nu(x) \to \varphi_\nu(x) - i\varphi_\nu(x) \star \lambda(x) + i\lambda(x) \star \varphi_\nu(x), \\ & \psi(x) \to \psi(x) - i\psi(x) \star \lambda(x) + i\lambda(x) \star \psi(x). \end{aligned} \tag{2.17}$$

3 Bi-local field representations

We have constructed our mapping rule eq.(2.8) by expanding matrices in terms of the wave functions $exp(ik \cdot \hat{x})$. It is the eigenstate of the adjoint $\hat{P}_\mu$ with the eigenvalue of k_μ. With the $n \times n$ matrix regularization, the momenta k_μ are quantized with a unit $k_{min} = \lambda n^{-1/\tilde{d}}$ and can take values $|k| < \lambda n^{1/\tilde{d}}/2$. These momentum eigenstates correspond to the ordinary plane waves when $|k_\mu| < \lambda$. In noncommutative space-time, it is not possible to consider states which are localized in the domain whose volume is smaller than the noncommutative scale. Therefore if we consider the states with larger momentum or smaller longitudinal length scale than the noncommutative scale, they must expand in the transverse directions. Recall that the both momentum space and coordinate space are embedded in the matrices of twisted reduced models. They are related by $\hat{x}^\mu = C^{\mu\nu}\hat{p}_\nu$. The corresponding eigenstates such as $exp(ik^1 \cdot \hat{x})$ and $exp(ik^2 \cdot \hat{x})$ are not commutative to each

other if $|k^i_\mu| > \lambda$, since $exp(ik^1 \cdot \hat{x})exp(ik^2 \cdot \hat{x}) = exp(ik^2 \cdot \hat{x})exp(ik^1 \cdot \hat{x})exp(iC^{\mu\nu}k^1_\mu k^2_\nu)$. The momentum eigenstate $exp(ik \cdot \hat{x})$ is also written as $exp(-id \cdot \hat{p})$ where $d^\mu = C^{\mu\nu}k_\nu$, and this implies that the eigenstate with $|k_\mu| > \lambda$ is more appropriately interpreted as noncommutative translation operators (2.10) rather than the ordinary plane waves. In other words, the eigenstate $exp(ik \cdot \hat{x})$ with $|k_\mu| > \lambda$ may be interpreted as string like extended objects whose length is $|C^{\mu\nu}k_\nu|$.

3.1 Operator - bi-local field mapping

In order to make these statements more transparent, we consider another representation of matrices in this section. For simplicity we consider the two dimensional case first:

$$[\hat{x}, \hat{y}] = -iC \tag{3.1}$$

where C is positive. This commutation relation is realized by the guiding center coordinates of the two dimensional system of electrons in magnetic field. The generalizations to arbitrary even $\tilde{d}$ dimensions are straightforward. We recall that we have n quanta with n dimensional matrices. Each quantum occupies the space-time volume of $2\pi C$. We may consider a square von Neumann lattice with the lattice spacing l_{NC} where $l^2_{NC} = 2\pi C$. This spacing l_{NC} gives the noncommutative scale. Let us denote the most localized state centered at the origin by $|0\rangle$. It is annihilated by the operator $\hat{x}^- = \hat{x} - i\hat{y}$. We construct states localized around each lattice site by utilizing translation operators $|x_i\rangle = exp(-ix_i \cdot \hat{p})|0\rangle$. They are the coherent states on a von Neumann lattice $\mathbf{x}_i = l_{NC}(n_i\mathbf{e}^x + m_i\mathbf{e}^y)$ where $n, m \in \mathbf{Z}$. They are complete but non-orthogonal. In the case of higher dimensions $\tilde{d}$, we introduce

$$\hat{x}^{\pm a} = \hat{x}^{2a-1} \pm i\hat{x}^{2a} \tag{3.2}$$

where $a = 1, \cdots, \tilde{d}/2$ and define the states $|\mathbf{x}_i\rangle$ accordingly. In the following, we set $C^{2a-1,2a} = C$ for simplicity.

The basic identity in this section is

$$\langle 0|exp(ik \cdot \hat{x})|0\rangle = exp(-\frac{Ck^2}{4}). \tag{3.3}$$

Using this identity, we indeed find

$$\rho_{ij} \equiv \langle x_i|x_j\rangle = exp(\frac{i}{2}B_{\mu\nu}x^\mu_i x^\nu_j)exp(-\frac{(x_i - x_j)^2}{4C}). \tag{3.4}$$

Although $|x_i\rangle$ are non-orthogonal, $\langle x_i|x_j\rangle$ exponentially vanishes when $(x_i - x_j)^2$ gets large. We note that the following wave functions of the two dimensional system of free electrons in the lowest Landau level are exponentially localized around x_i:

$$c_{n_i m_i}(x) = \frac{1}{l_{NC}} < x|x_i > . \tag{3.5}$$

Completeness of the basis leads to the resolution of unity

$$1 = \sum_{i,j} (\rho^{-1})_{ij} |x_i\rangle\langle x_j| \tag{3.6}$$

and the trace of an operator $\hat{\mathcal{O}}$ is given by

$$Tr\hat{\mathcal{O}} = \sum_{i,j} (\rho^{-1})_{ij} \langle x_j|\hat{\mathcal{O}}|x_i\rangle. \tag{3.7}$$

We also find

$$\langle x_i|exp(ik\cdot\hat{x})|x_j\rangle = exp(ik\cdot\frac{(x_i+x_j)}{2} + \frac{i}{2}B_{\mu\nu}x_i^\mu x_j^\nu)exp(-\frac{(x_i-x_j-d)^2}{4C}) \tag{3.8}$$

where $d^\mu = C^{\mu\nu}k_\nu$. This matrix element sharply peaks at $x_i - x_j = d$. It supports our interpretation that the eigenstate $exp(ik\cdot\hat{x})$ with $|k_\mu| > \lambda$ can be interpreted as string like extended objects whose length is $|C^{\mu\nu}k_\nu|$. When $|k_\mu| < \lambda$, on the other hand, this matrix becomes close to diagonal whose matrix elements go like

$$\langle x_i|exp(ik\cdot\hat{x})|x_j\rangle \sim exp(ik\cdot x_i)\langle x_i|x_j\rangle. \tag{3.9}$$

It again supports our interpretation that $exp(ik\cdot\hat{x})$ correspond to the ordinary plane waves when $|k_\mu| < \lambda$. They are represented by the matrices which are close to diagonal.

We now propose the bi-local field representation of noncommutative field theories. We may expand matrices $\hat{\phi}$ in the twisted reduced model by the following bi-local basis as follows:

$$\hat{\phi} = \sum_{i,j} \phi(x_i, x_j)|x_i\rangle\langle x_j| \tag{3.10}$$

where the Hermiticity of $\hat{\phi}$ implies $\phi^*(x_j, x_i) = \phi(x_i, x_j)$. The matrices $\hat{\phi}$ represent $\hat{a_\mu}$ or $\hat{\psi}$ in the super Yang-Mills case but the setting here is more generally applied to an arbitrary noncommutative field theory. The bi-local basis spans the whole n^2 degrees of freedom of matrices. The product of two operators is also given as

$$\hat{\phi}_1\hat{\phi}_2 = \sum_{i,j,k,l} \phi_1(x_i, x_j)\rho_{jk}\phi_2(x_k, x_l)|x_i\rangle\langle x_l| \tag{3.11}$$

and therefore

$$(\phi_1\phi_2)(x_i,x_j)=\sum_{k,l}\phi_1(x_i,x_k)\rho_{kl}\phi_2(x_l,x_j). \tag{3.12}$$

The trace of operators $\hat{\mathcal{O}}_1\cdots\hat{\mathcal{O}}_s$ in the bi-local basis is

$$Tr\hat{\mathcal{O}}_1\cdots\hat{\mathcal{O}}_s=\sum_{i_1\cdots i_{2s}}\mathcal{O}_1(x_{i_1},x_{i_2})\rho_{i_2i_3}\cdots\mathcal{O}_s(x_{i_{2s-1}},x_{i_{2s}})\rho_{i_{2s}i_1}. \tag{3.13}$$

Here we work out the translation rule between the momentum eigenstate representation $\hat{\phi}=\sum_k\tilde{\phi}(k)exp(ik\cdot\hat{x})$ and the bi-local field representation of eq.(3.10):

$$\begin{aligned}\tilde{\phi}(k) &= \frac{1}{n}Tr(exp(-ik\cdot\hat{x})\hat{\phi})=\frac{1}{n}\sum_{i,j}\langle x_i|exp(-ik\cdot\hat{x})|x_j\rangle\ \phi(x_j,x_i)\\ &= \frac{1}{n}\sum_{x_c}\phi(x_c)exp(-ik\cdot x_c),\\ \phi(x_c) &= \sum_{x_r}exp(\frac{i}{2}B_{\mu\nu}x_r^\mu x_c^\nu)exp(-\frac{(x_r-d)^2}{4C})\phi(x_j,x_i),\end{aligned} \tag{3.14}$$

where $x_c=(x_i+x_j)/2$ and $x_r=x_i-x_j$. From eq.(3.14), we observe that the slowly varying field with the momentum smaller than λ consists of the almost diagonal components. Hence close to diagonal components of the bi-local field are identified with the ordinary slowly varying field $\phi(x_c)$. On the other hand, rapidly oscillating fields are mapped to the off-diagonal open string states. A large momentum in ν direction $|k_\nu|>\lambda$ corresponds to a large distance in the μ-th direction $|d^\mu|=|C^{\mu\nu}k_\nu|>l_{NC}$. We can decompose d as $d=d_0+\delta d$ where d_0 is a vector which connects two points on the von Neumann lattice and $|\delta d|<l_{NC}$. Then the summation over x_r in (3.14) is dominated at $x_r=d_0$. In this way the large momentum degrees of freedom are more naturally interpreted as extended open string-like fields. They are denoted by 'open strings' in this paper.

The adjoint P^2 acts on $\hat{\phi}$ as

$$\begin{aligned}&P^2\hat{\phi}=\frac{B^2}{2}X^2\hat{\phi}\\ =\ &\frac{B^2}{2}(\hat{x}^{+a}\hat{x}^{-a}\hat{\phi}-\hat{x}^{-a}\hat{\phi}\hat{x}^{+a}-\hat{x}^{+a}\hat{\phi}\hat{x}^{-a}+\hat{\phi}\hat{x}^{+a}\hat{x}^{-a}+\tilde{d}C\hat{\phi}).\end{aligned} \tag{3.15}$$

The kinetic term of bosonic fields is of the form $-Tr((P_\mu\hat{\phi})^2)=Tr(\hat{\phi}P^2\hat{\phi})$ and is evaluated in this basis as

$$\begin{aligned}&\frac{1}{2}Tr\hat{\phi}P^2\hat{\phi}=\frac{B^2}{2}Tr\hat{\phi}X^2\hat{\phi}\\ =\ &\frac{B^2}{2}\sum_{i,j,k,l}\phi(x_i,x_j)\phi(x_k,x_l)\rho_{jk}\rho_{li}\{(x_l^{+a}-x_j^{+a})(x_i^{-a}-x_k^{-a})+\tilde{d}C\},\end{aligned} \tag{3.16}$$

where we have used the property of the coherent basis $\hat{x}^{-a}|x_i>= x_i^{-a}|x_i>$.

Here we decompose ρ_{ij} by inserting the identity constructed by the orthogonal basis:

$$1 = \sum_k |x_k\}\{x_k|. \tag{3.17}$$

An explicit construction of orthogonal localized wave functions $w_{m_in_i}(x) =< x|x_i\}$ is carried out in the two dimensional case[14]. They have a nearly Gaussian shape up to the radius $r < l_{NC}$ from the center. Outside this region, it is small, oscillates, and falls off with r^{-2}. The remarkable property of the asymptotic form is that it vanishes on the von Neumann lattice. It implies that $< x_i|x_j\}$ vanish for large $|x_i - x_j|$ very rapidly. Then eq.(3.16) can be rewritten for large $|x_i - x_j|$ as

$$\begin{aligned}
& \frac{B^2}{2}\sum_{i,j,k,l,n,m} \phi(x_i,x_j)\phi(x_k,x_l) < x_j|x_m\}\{x_m|x_k >< x_l|x_n\}\{x_n|x_i > \\
& \times(x_l^{+a} - x_j^{+a})(x_i^{-a} - x_k^{-a}) \\
= & \frac{B^2}{2}\sum_{n,m} \check{\phi}(x_n,x_m)\check{\phi}(x_m,x_n)(x_n - x_m)^2,
\end{aligned} \tag{3.18}$$

where

$$\check{\phi}(x_i,x_j) = \sum_{k,l}\{x_i|x_k > \phi(x_k,x_l) < x_l|x_j\} = \{x_i|\hat{\phi}|x_j\}. \tag{3.19}$$

In this derivation, we have used the fact that $\{x_n|x_i >$ are supported only for small $|x_n - x_i|$.

The kinetic term is diagonal for large $|x_i - x_j|$ and we read off the propagator of the bi-local field

$$\langle\check{\phi}(x_i,x_j)\check{\phi}(x_j,x_i)\rangle = \frac{C^2}{(x_i - x_j)^2}. \tag{3.20}$$

Long bi-local fields are thus interpreted to be far off-shell. It may be reinterpreted as

$$\langle\check{\phi}(p_i,p_j)\check{\phi}(p_j,p_i)\rangle = \frac{1}{(p_i - p_j)^2}, \tag{3.21}$$

where $p_{i,\mu} = B_{\mu\nu}x_i^{\nu}$. For small $|x_i - x_j|$, it is more appropriate to use the momentum eigenstate representation and we can obtain the standard propagator in that way.

We next consider three point vertices. Let us first consider the simplest three point vertex:

$$\begin{aligned}
Tr(\hat{\phi}^3) &= \sum_{i,j,k,l,m,n} \phi(x_i,x_j)\rho_{jk}\phi(x_k,x_l)\rho_{lm}\phi(x_m,x_n)\rho_{ni} \\
&= \sum_{i,j,k} \check{\phi}(x_i,x_j)\check{\phi}(x_j,x_k)\check{\phi}(x_k,x_i).
\end{aligned} \tag{3.22}$$

We can similarly obtain higher point vertices as well:

$$\begin{aligned} Tr(\hat{\phi}^s) &= \sum_{i_1\cdots i_{2s}} \phi(x_{i_1}, x_{i_2})\rho_{i_2 i_3}\cdots\phi(x_{i_{2s-1}}, x_{i_{2s}})\rho_{i_{2s}i_1} \\ &= \sum_{i_1\cdots i_s} \check{\phi}(x_{i_1}, x_{i_2})\cdots\check{\phi}(x_{i_s}, x_{i_1}). \end{aligned} \tag{3.23}$$

In this way, we make contact with the quenched reduced models [11] in the large momentum region. In quenched reduced models, the eigenvalues of matrices are identified with momenta. Therefore we have found that the large momentum behavior of twisted reduced models is identical to quenched reduced models. In this correspondence, the relative distance of the two ends of 'open string' in twisted reduced model is related to the momentum in quenched reduced model by the relation $\hat{p}_\mu = B_{\mu\nu}\hat{x}^\nu$.

3.2 Perturbations

Perturbative behaviors of noncommutative field theories have been investigated [15, 16, 17, 18, 19, 20, 21]. In particular, it was pointed out that the effective action exhibits infrared singular behaviors due to the nonplanar diagrams[19, 21]. We now look at the perturbation of the noncommutative field theories in the bi-local basis. Let us consider the ϕ^3 theory as a simple example. The matrix model action is given by

$$S = Tr\left(-\frac{1}{2}[\hat{p}_\mu, \hat{\phi}]^2 + \frac{\lambda}{3}\hat{\phi}^3\right). \tag{3.24}$$

In the large momentum region, the propagator and vertex are given by (3.20) and (3.22). In the one-loop approximation, there are two types of diagrams.

Fig. 1 is planar and gives the correction to the propagator of the bi-local field $\check{\phi}(x_i, x_j)$:

$$\sum_{i,j}\sum_k \frac{C^4\check{\phi}(x_i, x_j)\check{\phi}(x_j, x_i)}{(x_i - x_k)^2(x_j - x_k)^2}. \tag{3.25}$$

This diagram corresponds to the following planar diagram amplitude in the conventional perturbative expansion of noncommutative field theories:

$$\sum_{p,k} \frac{\tilde{\phi}(p)\tilde{\phi}(-p)}{k^2(p-k)^2}. \tag{3.26}$$

As originally proved in [22], noncommutative phases which can be assigned to the propagators are canceled in planar diagrams and the amplitudes of such graphs are the same as the commutative cases except for the phase factors which only depend on the external momenta.

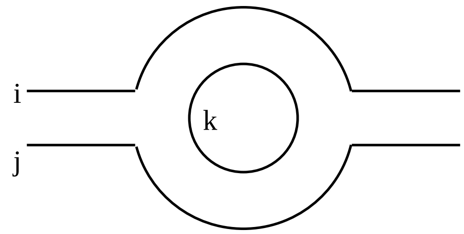

Figure 1: A planar graph at one loop: it renormalizes the kinetic term of bi-local fields $\check{\phi}(x_i, x_j)$.

In $\tilde{d}$-dimensions, the integral (3.26) is UV divergent as $\Lambda^{\tilde{d}-4}$ where Λ is the ultraviolet cutoff. In the bi-local representation (3.25), the summation is also divergent but it originates from the infrared extension of the von Neumann lattice. It is reminiscent of the tadpole divergences in string theory. However this analogy requires more work to substantiate it. For example, we may be able to work out a direct world sheet interpretation of the bi-local propagators.

Fig. 2 is a nonplanar diagram and induces effective interactions for only the diagonal components $\check{\phi}(x_i, x_i)$ of the bi-local fields. Since the diagonal components are interpreted as ordinary (slowly varying) local fields $\phi(x_i)$, this diagram induces long range interactions:

$$S_{eff} = C^4 \sum_{i,j} \frac{\phi(x_i)\phi(x_j)}{(x_i - x_j)^4}. \tag{3.27}$$

This type of interactions have been well known as the block-block interactions in the matrix models [1]. In the conventional picture of noncommutative ϕ^3 theory, this diagram provides

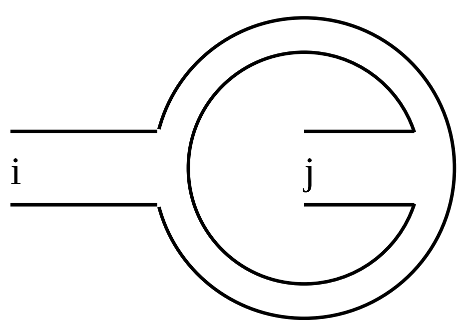

Figure 2: A nonplanar graph at one loop: it generates long range interaction between diagonal components of the bi-local field $\check{\phi}(x_i, x_i)$.

the following nonplanar one-loop correction to the propagator.

$$\sum_{p,k} \frac{\tilde{\phi}(p)\tilde{\phi}(-p)}{k^2(p-k)^2} exp(iC_{\mu\nu}k^\mu p^\nu). \tag{3.28}$$

It is very specific to noncommutative field theory since it exhibits infrared singular behaviors for small p.

A physically intuitive argument which relates the nonplanar amplitude in the bi-local basis (3.27) and the conventional nonplanar amplitude (3.28) is given as follows:

$$\begin{aligned} S_{eff} &= C^4 \sum_{p,q}\sum_{i,j} \frac{\tilde{\phi}(p)\tilde{\phi}(q)}{(x_i - x_j)^4} e^{ip\cdot x_i} e^{iq\cdot x_j} \\ &= nC^4 \sum_{p}\sum_{d} \frac{\tilde{\phi}(p)\tilde{\phi}(-p)}{d^4} e^{ip\cdot d}. \end{aligned} \tag{3.29}$$

If we rewrite $k_\mu = B_{\mu\nu}d^\nu$, this has the same expression as eq.(3.28). Here we also recall that the momentum lattice $\{k_\mu\}$ has finer resolutions than the von Neumann lattice $\{d_\mu\}$. Therefore $\sum_k = n\sum_d$ for large $|d|$. Note that the nonplanar phase is interpreted as the wave functions of plane waves here and the summation over k in (3.28) amounts to performing the Fourier transformation. On the other hand, the summation over k in (3.26) leads to the standard one loop integration in the planar contribution.

From these arguments, it is clear that the long range interaction $1/d^4$ is still induced even when the ϕ field has a mass term m as long as $d > Cm$. In the next section, we investigate these long range interactions of matrix models in more detail.

4 Interactions between diagonal blocks

In this section, we investigate the renormalization property of noncommutative field theory. After integrating long 'open strings', we obtain long range interactions. This phenomena appear as nonplanar contributions in noncommutative field theory[19]. We show that it can be simply understood as block-block interactions in the matrix model picture.

4.1 Noncommutative ϕ^3

We consider a noncommutative ϕ^3 field theory as a simple example. The matrix model action is given by (3.24). Following the same procedure as in eqs. (2.5 - 2.8), we can obtain a noncommutative $U(m)$ ϕ^3 field theory:

$$S = \int d^{\tilde{d}}x \; tr \left(\frac{1}{2}(\partial_\mu \phi(x))^2 + \frac{\lambda'}{3}\phi(x)^3 \right)_\star . \tag{4.1}$$

Here tr means a trace over $(m \times m)$ matrices and $\lambda' = \lambda(2\pi/B)^{\tilde{d}/4}$.

We consider a group of backgrounds $\phi^{(i)}$. We assume that the corresponding functions $\phi^{(i)}(x)$ are localized and well separated. $\phi^{(i)}$ can be specified by its Fourier components $\tilde{\phi}^{(i)}(k)$:

$$\hat{\phi}^{(i)} = \sum_k \tilde{\phi}^{(i)}(k) exp(ik \cdot \hat{x}). \tag{4.2}$$

We further assume that $\tilde{\phi}^{(i)}(k)$ is only supported for $|k| << \lambda$ which implies the much larger localization scale than the noncommutativity scale. This condition means that the background in the bi-local representation $\phi^{(i)}(x, y)$ is almost diagonal. We hence expect that the following representation is accurate :

$$\hat{\phi}^{(i)} = \sum_j \phi^{(i)}(x_j)|x_j\rangle\langle x_j|. \tag{4.3}$$

The background is assumed to be localized around $x = d^{(i)}$. In this section we obtain the effective action for these backgrounds by integrating over the off-diagonal components of the bi-local field, that is, the higher momentum fields.

The assumption of no overlap imposes certain conditions on $\tilde{\phi}^{(i)}(k)$. The commutators are mapped in our mapping rule as follows:

$$[\phi^{(i)}, \phi^{(j)}] \to \phi^{(i)}(x) \star \phi^{(j)}(x) - \phi^{(j)}(x) \star \phi^{(i)}(x). \tag{4.4}$$

Then we can see that $\phi^{(i)}$ and $\phi^{(j)}$ are commutative to each other from eq.(4.4). Therefore they can be represented by a block diagonal form as follows:

$$\hat{\phi}_{cl} = \sum \hat{\phi}^{(i)} = \begin{pmatrix} \phi^{(1)} & & & \\ & \phi^{(2)} & & \\ & & \phi^{(3)} & \\ & & & \ddots \end{pmatrix}, \tag{4.5}$$

where $\phi^{(i)}$ $(i = 1, 2, \cdots)$ is a $n_i \times n_i$ matrix.

$[\hat{p}_\mu, \phi^{(i)}]$ can be mapped onto $-i\partial_\mu \phi^{(i)}(x)$. Since $\partial_\mu \phi^{(i)}(x) \star \phi^{(j)}(x) - \phi^{(j)}(x) \star \partial_\mu \phi^{(i)}(x)$ vanishes for well separated wave-packets, $[\hat{p}_\mu, \phi^{(i)}]$ are also of the block diagonal form as in eq.(4.5). This fact in turn implies that $\hat{p}_\mu$ can be represented in the same block diagonal form when it acts on these backgrounds:

$$\hat{p}_\mu = \begin{pmatrix} p_\mu^{(1)} & & & \\ & p_\mu^{(2)} & & \\ & & p_\mu^{(3)} & \\ & & & \ddots \end{pmatrix}, \tag{4.6}$$

Here $p_\mu^{(i)}$ denotes the projection of $\hat{p}_\mu$ onto the subspace specified by $\phi^{(i)}$. $\hat{p}_\mu$ can be represented as the block diagonal form since we have projected it onto the subspace where the backgrounds $\phi^{(i)}$ are supported.

We now calculate the one-loop effective action between diagonal blocks in order to investigate the renormalization property of noncommutative field theory. We may further decompose $p_\mu^{(i)}$ as

$$\begin{aligned} p_\mu^{(i)} &= B_{\mu\nu} d_\nu^{(i)} 1_{n_i} + \tilde{p}_\mu^{(i)}, \\ Tr \tilde{p}_\mu^{(i)} &= 0, \end{aligned} \tag{4.7}$$

where $d_\nu^{(i)}$ is a c number representing the center of mass coordinate of the i-th block. Here we assume that the blocks are separated far enough from each other, that is, for all i and j's, $(d_\mu^{(i)} - d_\mu^{(j)})^2$'s are larger than the localization scale of each block.

We first recall some notations which are introduced in [1]. We denote the (i,j) block of a matrix X as $X^{(i,j)}$. It is clear that $P_\mu = [\hat{p}_\mu, \ ]$ operates on each $X^{(i,j)}$ independently. In fact we have

$$(P_\mu X)^{(i,j)} = B_{\mu\nu}(d_\nu^{(i)} - d_\nu^{(j)}) X^{(i,j)} + \tilde{p}_\mu^{(i)} X^{(i,j)} - X^{(i,j)} \tilde{p}_\mu^{(j)}. \tag{4.8}$$

In the bi-local basis developed in the previous section, (i,j) block of a matrix $X^{(i,j)}$ may correspond to a collections of bi-local fields which connect the i-th and j-th diagonal blocks. We further simplify this equation by introducing notations such as

$$\begin{aligned} d_\mu^{(i,j)} X^{(i,j)} &= (d_\mu^{(i)} - d_\mu^{(j)}) X^{(i,j)}, \\ P_{L\mu}^{(i,j)} X^{(i,j)} &= \tilde{p}_\mu^{(i)} X^{(i,j)}, \\ P_{R\mu}^{(i,j)} X^{(i,j)} &= -X^{(i,j)} \tilde{p}_\mu^{(j)}. \end{aligned} \tag{4.9}$$

Note that $d_\mu^{(i,j)}$,$P_{L\mu}^{(i,j)}$ and $P_{R\mu}^{(i,j)}$ commute each other, and the operation of P_μ on $X^{(i,j)}$ is expressed as

$$(P_\mu X)^{(i,j)} = (B_{\mu\nu} d_\nu^{(i,j)} + P_{L\mu}^{(i,j)} + P_{R\mu}^{(i,j)}) X^{(i,j)}. \tag{4.10}$$

We can also decompose the action of $\hat{\phi}_{cl}$ onto $X^{(i,j)}$, which should be made symmetric between the right and the left multiplications for Hermiticity, in the same way as (4.10):

$$(\Phi X)^{(i,j)} \equiv (\Phi_L^{(i,j)} + \Phi_R^{(i,j)}) X^{(i,j)}, \tag{4.11}$$

where

$$\begin{aligned} \Phi_L^{(i,j)} X^{(i,j)} &= \phi^{(i)} X^{(i,j)}, \\ \Phi_R^{(i,j)} X^{(i,j)} &= X^{(i,j)} \phi^{(j)}. \end{aligned} \tag{4.12}$$

Since the left and right multiplication are totally independent, we have

$$
\begin{aligned}
(OX)^{(i,j)} &\equiv O_L^{(i,j)} X^{(i,j)} O_R^{(i,j)}, \\
\mathcal{T}r\, O &= \sum_{i,j=1}^{n} Tr O_L^{(i,j)} Tr O_R^{(i,j)},
\end{aligned} \tag{4.13}
$$

for operators consisting of P_μ and Φ. Here $\mathcal{T}r$ denotes the trace of the operators which act on the matrices.

The one loop effective action is

$$
W = \frac{1}{2}\mathcal{T}r log(1 - \frac{1}{P_\mu^2}\lambda\Phi)). \tag{4.14}
$$

Now we expand the expression of the one-loop effective action with respect to the inverse power of $d_\mu^{(i,j)}$'s. We drop the linear term in the background fields since we adopt the background field method which has no ambiguity in this case unlike the gauge fixing ambiguity in gauge theory. We have,

$$
\begin{aligned}
W &= -\frac{1}{4}\mathcal{T}r\left(\frac{1}{P^2}\lambda\Phi\frac{1}{P^2}\lambda\Phi\right) \\
&\quad -\frac{1}{6}\mathcal{T}r\left(\frac{1}{P^2}\lambda\Phi\frac{1}{P^2}\lambda\Phi\frac{1}{P^2}\lambda\Phi\right) \\
&\quad -\frac{1}{8}\mathcal{T}r\left(\frac{1}{P^2}\lambda\Phi\frac{1}{P^2}\lambda\Phi\frac{1}{P^2}\lambda\Phi\frac{1}{P^2}\lambda\Phi\right) + O((\Phi)^5).
\end{aligned} \tag{4.15}
$$

Since as in (4.10) and (4.11) P_μ and Φ act on the (i,j) blocks independently, the one-loop effective action W is expressed as the sum of contributions of the (i,j) blocks $W^{(i,j)}$. Therefore we may consider $W^{(i,j)}$ as the interaction between the i-th and j-th blocks. Using (4.13) and (4.15) we can easily evaluate $W^{(i,j)}$ to the leading order of $1/\sqrt{(d^{(i)} - d^{(j)})^2}$ as

$$
\begin{aligned}
W^{(i,j)} &= \frac{C^4}{(d^{(i)} - d^{(j)})^4}(-\frac{\lambda^2}{4})\mathcal{T}r^{(i,j)}\left(\Phi\Phi\right) \\
&\quad + O((1/(d^{(i)} - d^{(j)})^5) \\
&= \frac{C^4}{(d^{(i)} - d^{(j)})^4}(-\frac{\lambda^2}{4})(n_j Tr(\phi^{(i)}\phi^{(i)}) + n_i Tr(\phi^{(j)}\phi^{(j)}) + 2Tr(\phi^{(i)})Tr(\phi^{(j)})) \\
&\quad + O((1/(d^{(i)} - d^{(j)})^5).
\end{aligned} \tag{4.16}
$$

The third term in the above expression can be interpreted as the signature for the existence of massless particles corresponding to a scalar in six dimensions.

A more conventional approach to calculate $\mathcal{T}r$ is to use the plane-wave basis $exp(ik\cdot\hat{x})$. It corresponds to a standard one loop calculation in noncommutative ϕ^3:

$$
W = -\frac{1}{4}\mathcal{T}r\left(\frac{1}{P^2}\lambda\Phi\frac{1}{P^2}\lambda\Phi\right)
$$

$$= -\frac{1}{4n}\sum_k Tr(exp(-ik\cdot\hat{x})\frac{1}{P^2}\lambda\Phi\frac{1}{P^2}\lambda\Phi exp(ik\cdot\hat{x}))$$
$$= -\frac{\lambda^2}{4}\sum_l \tilde{\phi}_{cl}(-l)\tilde{\phi}_{cl}(l)\sum_k \frac{2}{k^2(k+l)^2}(1+exp(iC^{\mu\nu}k_\mu l_\nu)). \tag{4.17}$$

Here we find the both planar and nonplanar contributions. The latter contains the nontrivial phase factor $exp(iC^{\mu\nu}k_\mu l_\nu)$.

We evaluate the nonplanar contributions:

$$\frac{1}{n}\sum_k \frac{1}{k^2(k+l)^2}exp(iC^{\mu\nu}k_\mu l_\nu)$$
$$= (\frac{1}{2\pi B})^{\frac{\tilde{d}}{2}}\int d^{\tilde{d}}k\frac{1}{k^2(k+l)^2}exp(iC^{\mu\nu}k_\mu l_\nu)$$
$$= (\frac{1}{2\pi B})^{\frac{\tilde{d}}{2}}\int d\alpha_1 d\alpha_2(\frac{\pi}{\alpha_1+\alpha_2})^{\frac{\tilde{d}}{2}}exp(-\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}l^2-\frac{(Cl)^2}{4(\alpha_1+\alpha_2)}). \tag{4.18}$$

When $\tilde{d}>4$, the above integral is evaluated as

$$\Gamma(\frac{\tilde{d}}{2}-2)(\frac{C}{2})^{4-\frac{\tilde{d}}{2}}(l^2)^{2-\frac{\tilde{d}}{2}}. \tag{4.19}$$

We therefore find the nonplanar contribution:

$$-n\frac{\lambda^2}{2}\sum_l \tilde{\phi}(-l)\tilde{\phi}(l)(\frac{C}{2})^{4-\frac{\tilde{d}}{2}}(l^2)^{2-\frac{\tilde{d}}{2}}\Gamma(\frac{\tilde{d}}{2}-2)$$
$$= -n^2\frac{\lambda^2C^4}{2}(\frac{1}{2})^{4-\frac{\tilde{d}}{2}}\Gamma(\frac{\tilde{d}}{2}-2)(\frac{1}{2\pi})^{\frac{\tilde{d}}{2}}\int d^{\tilde{d}}l\tilde{\phi}(-l)\tilde{\phi}(l)(l^2)^{2-\frac{\tilde{d}}{2}}$$
$$= -\frac{\lambda^2C^4}{2}(\frac{1}{2\pi C})^{\tilde{d}}\int d^{\tilde{d}}x\int d^{\tilde{d}}y\phi(x)\frac{1}{(x-y)^4}\phi(y) \tag{4.20}$$

where we have used the identity:

$$\int\frac{d^{\tilde{d}}k}{(2\pi)^{\tilde{d}}}exp(ik\cdot x)(\frac{1}{k^2})^{\frac{\tilde{d}}{2}-2} = \frac{1}{2^{\tilde{d}-4}}(\frac{1}{\pi})^{\frac{\tilde{d}}{2}}\frac{1}{\Gamma(\frac{\tilde{d}}{2}-2)}\frac{1}{|x|^4}. \tag{4.21}$$

We also note that our convention is

$$\tilde{\phi}_{cl}(l) = \frac{1}{n(2\pi C)^{\frac{\tilde{d}}{2}}}\int d^{\tilde{d}}xexp(-il\cdot x)\phi_{cl}(x). \tag{4.22}$$

We observe that eq.(4.20) can be identified with the last term of eq.(4.16) since it can be reexpressed as follows due to our mapping rule eq.(2.8):

$$\sum_{ij}\frac{C^4}{(d^{(i)}-d^{(j)})^4}(-\frac{\lambda^2}{2})Tr(\phi^{(i)})Tr(\phi^{(j)})$$
$$= -\frac{\lambda^2C^4}{2}(\frac{1}{2\pi C})^{\tilde{d}}\int d^{\tilde{d}}x\int d^{\tilde{d}}ytr\phi_{cl}(x)\frac{1}{(x-y)^4}tr\phi_{cl}(y). \tag{4.23}$$

Here we have replaced the trace over the localized classical background ϕ_i by the integration over the space-time coordinates in the neighborhood of the wave packet. By integrating over the entire space-time, we also sum over different localized blocks. Since we are only considering the interactions of the well separated backgrounds, the above formula is only valid in the low momentum region. What we have found here is that the nonplanar contributions which give rise to the long range interactions can be identified with those from off-diagonal components in matrix models. We can simply reproduce it by considering block-block interactions. These are distinguished features of noncommutative field theory which are absent in field theory.

4.2 Noncommutative ϕ^4

We consider noncommutative ϕ^4 theory next

$$S = Tr\left(-\frac{1}{2}[\hat{p}_\mu, \hat{\phi}]^2 + \frac{\lambda}{4}\hat{\phi}^4\right). \tag{4.24}$$

The one loop effective action is

$$W = \frac{1}{2}\mathcal{T}rlog(1 - \frac{1}{P_\mu^2}\lambda(\Phi_L^2 + \Phi_R^2 + \Phi_L\Phi_R)). \tag{4.25}$$

Now we expand the expression of the one-loop effective action with respect to the inverse power of $d_\mu^{(i,j)}$'s. We have,

$$\begin{aligned} W &= -\frac{1}{2}\mathcal{T}r\left(\frac{1}{P^2}\lambda(\Phi_L^2 + \Phi_R^2 + \Phi_L\Phi_R)\right) \\ &\quad -\frac{1}{4}\mathcal{T}r\left(\frac{1}{P^2}\lambda(\Phi_L^2 + \Phi_R^2 + \Phi_L\Phi_R)\frac{1}{P^2}\lambda(\Phi_L^2 + \Phi_R^2 + \Phi_L\Phi_R)\right) + O((\Phi)^5). \end{aligned} \tag{4.26}$$

We can easily evaluate $W^{(i,j)}$ to the leading order of $1/\sqrt{(d^{(i)} - d^{(j)})^2}$ as

$$\begin{aligned} W^{(i,j)} &= \frac{1}{(d^{(i)} - d^{(j)})^2}(-\frac{\lambda}{2})\mathcal{T}r^{(i,j)}\left(\Phi_L^2 + \Phi_R^2 + \Phi_L\Phi_R\right) \\ &\quad +O((1/(d^{(i)} - d(j))^3) \\ &= \frac{1}{(d^{(i)} - d^{(j)})^2}(-\frac{\lambda}{2})(n_jTr(\phi^{(i)}\phi^{(i)}) + n_iTr(\phi^{(j)}\phi^{(j)}) + Tr(\phi^{(i)})Tr(\phi^{(j)})) \\ &\quad +O((1/(d^{(i)} - d^{(j)})^3). \end{aligned} \tag{4.27}$$

We find the exchanges of massless particles corresponding to a scalar in four dimensions.

We also evaluate the leading term of the effective action by using the plane-wave basis $exp(ik\cdot\hat{x})$.

$$\begin{aligned} W &= -\frac{1}{2}\mathcal{T}r\left(\frac{1}{P^2}\lambda(\Phi_L^2+\Phi_R^2+\Phi_L\Phi_R)\right) \\ &= -\frac{1}{2n}\sum_k Tr(exp(-ik\cdot\hat{x})\frac{1}{P^2}\lambda(\Phi_L^2+\Phi_R^2+\Phi_L\Phi_R)exp(ik\cdot\hat{x})) \\ &= -\frac{\lambda}{2}\sum_l\tilde{\phi}(-l)\tilde{\phi}(l)\sum_k\frac{1}{k^2}(2+exp(iC^{\mu\nu}k_\mu l_\nu)). \end{aligned} \tag{4.28}$$

Here we also find the both planar and nonplanar contributions.

We evaluate the nonplanar contributions:

$$\begin{aligned} &\frac{1}{n}\sum_k\frac{1}{k^2}exp(iC^{\mu\nu}k_\mu l_\nu) \\ =\ &(\frac{1}{2\pi B})^{\frac{\tilde{d}}{2}}\int d^{\tilde{d}}k\frac{1}{k^2}exp(iC^{\mu\nu}k_\mu l_\nu) \\ =\ &(\frac{1}{2\pi B})^{\frac{\tilde{d}}{2}}\int d\alpha(\frac{\pi}{\alpha})^{\frac{\tilde{d}}{2}}exp(-\frac{(Cl)^2}{4\alpha}). \end{aligned} \tag{4.29}$$

When $\tilde{d}>2$, the above integral is evaluated as

$$\Gamma(\frac{\tilde{d}}{2}-1)(\frac{C}{2})^{2-\frac{\tilde{d}}{2}}(l^2)^{1-\frac{\tilde{d}}{2}}. \tag{4.30}$$

We therefore find the nonplanar contribution:

$$\begin{aligned} &-n\frac{\lambda}{2}\sum_l\tilde{\phi}(-l)\tilde{\phi}(l)(\frac{C}{2})^{2-\frac{\tilde{d}}{2}}(l^2)^{1-\frac{\tilde{d}}{2}}\Gamma(\frac{\tilde{d}}{2}-1) \\ =\ &-n^2\frac{\lambda C^2}{2}(\frac{1}{2})^{2-\frac{\tilde{d}}{2}}\Gamma(\frac{\tilde{d}}{2}-1)(\frac{1}{2\pi})^{\frac{\tilde{d}}{2}}\int d^{\tilde{d}}l\tilde{\phi}(-l)\tilde{\phi}(l)(l^2)^{1-\frac{\tilde{d}}{2}} \\ =\ &-\frac{\lambda C^2}{2}(\frac{1}{2\pi C})^{\tilde{d}}\int d^{\tilde{d}}x\int d^{\tilde{d}}y\phi(x)\frac{1}{(x-y)^2}\phi(y) \end{aligned} \tag{4.31}$$

where we have used the identity:

$$\int\frac{d^{\tilde{d}}k}{(2\pi)^{\tilde{d}}}exp(ik\cdot x)(\frac{1}{k^2})^{\frac{\tilde{d}}{2}-1}=\frac{1}{2^{\tilde{d}-2}}(\frac{1}{\pi})^{\frac{\tilde{d}}{2}}\frac{1}{\Gamma(\frac{\tilde{d}}{2}-1)}\frac{1}{|x|^2}. \tag{4.32}$$

We observe that eq.(4.31) can be identified with the last term of eq.(4.27) since it can be reexpressed as follows due to our mapping rule:

$$\begin{aligned} &\sum_{ij}\frac{C^2}{(d^{(i)}-d^{(j)})^2}(-\frac{\lambda}{2})Tr(\phi^{(i)})Tr(\phi^{(j)}) \\ =\ &-\frac{\lambda C^2}{2}(\frac{1}{2\pi C})^{\tilde{d}}\int d^{\tilde{d}}x\int d^{\tilde{d}}ytr\phi_{cl}(x)\frac{1}{(x-y)^2}tr\phi_{cl}(y). \end{aligned} \tag{4.33}$$

5 Block-block interactions in gauge theory

In the previous section we have seen that the long range block-block interactions are induced in the effective action for the slowly varying fields of noncommutative scalar theories after integrating out the off-diagonal components. They correspond to nonplanar diagrams which carry nontrivial phase factors.

These are the familiar interactions in matrix models and indeed in IIB matrix model they are interpreted as massless particle propagations [1]. There is a conceptual difference, however, between scalar theory and gauge theory. In scalar theory we must introduce extra matrices $\hat{p}_\mu$ to represent the noncommutative space-time. In reduced models of gauge theory, they are considered as special backgrounds of the dynamical variables ($A_\mu = \hat{p}_\mu + \hat{a}_\mu$). The block-block interactions in reduced models of gauge theory such as type IIB matrix model are universal and not restricted to specific backgrounds $\hat{p}_\mu$ which define the twisted reduced models. Actually the original calculation of such interactions in [1] does not assume any specific conditions for backgrounds. We have obtained the long range interactions which decay as $1/r^8$. It does not depend whether we assume uniform distributions of the matrix eigenvalues in ten dimensions or in lower dimensions such as four as is discussed in this section. These long range interactions are interpreted as propagations of the massless type IIB supergravity multiplets and this fact can be considered as one of the evidences that IIB matrix model contains gravity.

In this section we reexamine the block-block interactions of D-instantons in IIB matrix model[6]. There we have considered the backgrounds which represent four dimensional noncommutative space-time. By expanding IIB matrix model around such backgrounds, we obtain four dimensional noncommutative Yang-Mills theory. There are nontrivial classical solutions in this theory which reduce to instantons in the large instanton size limit[24].

We have considered a classical solution of IIB matrix model which represents an instanton and an (anti)instanton. We can realize $U(4)$ gauge theory by considering four D3 branes. We embed an instanton into the first $SU(2)$ part and the other (anti)instanton into the remaining $SU(2)$ part. We separate them in the fifth dimension by the distance b:

$$\begin{aligned}
A_0 &= \begin{pmatrix} p_0 + a_0 & 0 \\ 0 & p_0 + a_0' \end{pmatrix} \\
A_1 &= \begin{pmatrix} p_1 + a_1 & 0 \\ 0 & p_1 + a_1' \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
A_2 &= \begin{pmatrix} p_2 + a_2 & 0 \\ 0 & p_2 + a_2' \end{pmatrix} \\
A_3 &= \begin{pmatrix} p_3 + a_3 & 0 \\ 0 & p_3 + a_3' \end{pmatrix} \\
A_4 &= \begin{pmatrix} \frac{b}{2} & 0 \\ 0 & -\frac{b}{2} \end{pmatrix} \\
A_\rho &= 0
\end{aligned} \tag{5.1}
$$

where $\rho = 5, \cdots, 9$.

While two instanton system receives no quantum corrections, the instanton - anti-instanton system receives quantum corrections since it is no longer BPS. We have evaluated the one loop effective potential due to an instanton and (anti)instanton. They are local excitations and couple to gravity. These solutions are characterized by the adjoint field strength $F_{\mu\nu}$ which does not vanish at the locations of the instantons. We assume that they are separated by a long distance compared to their sizes. We also assume that $b >> l_{NC}$. Then we can choose two disjoint blocks in each of which a large part of an (anti)instanton is contained. Let the location and the size of the two instantons (x_i, ρ_i) and (x_j, ρ_j).[4] The ten dimensional distance of them is $r^2 = (x_i - x_j)^2 + b^2$. Here we have assumed that $r >> \rho$.

The one-loop effective action of IIB matrix model is

$$
ReW = \frac{1}{2}\mathcal{T}r \log(P_\lambda^2 \delta_{\mu\nu} - 2iF_{\mu\nu}) - \frac{1}{4}\mathcal{T}r \log((P_\lambda^2 + \frac{i}{2}F_{\mu\nu}\Gamma^{\mu\nu})(\frac{1+\Gamma_{11}}{2})) - \mathcal{T}r \log(P_\lambda^2). \tag{5.2}
$$

Here P_μ and $F_{\mu\nu}$ are operators acting on the space of matrices as

$$
\begin{aligned}
P_\mu X &= [p_\mu, X], \\
F_{\mu\nu} X &= [f_{\mu\nu}, X],
\end{aligned} \tag{5.3}
$$

where $f_{\mu\nu} = i[p_\mu, p_\nu]$. Now we expand the general expression of the one-loop effective action (5.2) with respect to the inverse power of $d_\mu^{(i,j)}$'s just like the preceding section. We can take traces of the γ matrices after expanding the logarithm in (5.2). Due to the supersymmetry, contributions of bosons and fermions cancel each other to the third order in $F_{\mu\nu}$, and we have,

$$
\begin{aligned}
W &= -\mathcal{T}r\left(\frac{1}{P^2}F_{\mu\nu}\frac{1}{P^2}F_{\nu\lambda}\frac{1}{P^2}F_{\lambda\rho}\frac{1}{P^2}F_{\rho\mu}\right) \\
&\quad -2\mathcal{T}r\left(\frac{1}{P^2}F_{\mu\nu}\frac{1}{P^2}F_{\lambda\rho}\frac{1}{P^2}F_{\mu\rho}\frac{1}{P^2}F_{\lambda\nu}\right)
\end{aligned}
$$

[4]Here we have used the indices i and j to denote two instantons since they are also represented by the diagonal blocks in the matrix model. However we consider only two instantons in this section.

$$+\frac{1}{2}\mathcal{T}r\left(\frac{1}{P^2}F_{\mu\nu}\frac{1}{P^2}F_{\mu\nu}\frac{1}{P^2}F_{\lambda\rho}\frac{1}{P^2}F_{\lambda\rho}\right)$$
$$+\frac{1}{4}\mathcal{T}r\left(\frac{1}{P^2}F_{\mu\nu}\frac{1}{P^2}F_{\lambda\rho}\frac{1}{P^2}F_{\mu\nu}\frac{1}{P^2}F_{\lambda\rho}\right)+O((F_{\mu\nu})^5). \quad (5.4)$$

Since as in (4.10) and (4.11) P_μ and $F_{\mu\nu}$ act on the (i,j) blocks independently, the one-loop effective action W is expressed as the sum of contributions of the (i,j) blocks $W^{(i,j)}$. Therefore we may consider $W^{(i,j)}$ as the interaction between the i-th and j-th blocks.

Using (4.13) and (5.4) we can easily evaluate $W^{(i,j)}$ to the leading order of $1/\sqrt{(d^{(i)}-d^{(j)})^2}$ as

$$\begin{aligned}
W^{(i,j)} &= \frac{1}{r^8}(-\mathcal{T}r^{(i,j)}(F_{\mu\nu}F_{\nu\lambda}F_{\lambda\rho}F_{\rho\mu})-2\mathcal{T}r^{(i,j)}(F_{\mu\nu}F_{\lambda\rho}F_{\mu\rho}F_{\lambda\nu})\\
&\quad+\frac{1}{2}\mathcal{T}r^{(i,j)}(F_{\mu\nu}F_{\mu\nu}F_{\lambda\rho}F_{\lambda\rho})+\frac{1}{4}\mathcal{T}r^{(i,j)}(F_{\mu\nu}F_{\lambda\rho}F_{\mu\nu}F_{\lambda\rho})\\
&= \frac{3}{2r^8}(-n_j\tilde{b}_8(f^{(i)})-n_i\tilde{b}_8(f^{(j)})\\
&\quad-8Tr(f^{(i)}_{\mu\nu}f^{(i)}_{\nu\sigma})Tr(f^{(j)}_{\mu\rho}f^{(j)}_{\rho\ sigma})+Tr(f^{(i)}_{\mu\nu}f^{(i)}_{\mu\nu})Tr(f^{(j)}_{\rho\sigma}f^{(j)}_{\rho\sigma})\\
&\quad+Tr(f^{(i)}_{\mu\nu}\tilde{f}^{(i)}_{\mu\nu})Tr(f^{(j)}_{\rho\sigma}\tilde{f}^{(j)}_{\rho\sigma})) \qquad (5.5)
\end{aligned}$$

where $\tilde{f}_{\mu\nu}=\epsilon_{\mu\nu\rho\sigma}f_{\rho\sigma}/2$ and

$$\tilde{b}_8(f)=\frac{2}{3}(Tr(f_{\mu\nu}f_{\nu\lambda}f_{\lambda\rho}f_{\rho\mu})+2Tr(f_{\mu\nu}f_{\lambda\rho}f_{\mu\rho}f_{\lambda\nu})-\frac{1}{2}Tr(f_{\mu\nu}f_{\mu\nu}f_{\lambda\rho}f_{\lambda\rho})-\frac{1}{4}Tr(f_{\mu\nu}f_{\lambda\rho}f_{\mu\nu}f_{\lambda\rho})). \quad (5.6)$$

In eq.(5.5), we have kept axion type interactions also. Note that the $\tilde{b}_8(f)=0$ for an (anti)instanton configuration. So the potential between an instanton and an (anti)instanton is

$$\frac{3}{2r^8}(-8Tr(f^{(i)}_{\mu\nu}f^{(i)}_{\nu\sigma})Tr(f^{(j)}_{\mu\rho}f^{(j)}_{\rho\sigma})+Tr(f^{(i)}_{\mu\nu}f^{(i)}_{\mu\nu})Tr(f^{(j)}_{\rho\sigma}f^{(j)}_{\rho\sigma})+Tr(f^{(i)}_{\mu\nu}\tilde{f}^{(i)}_{\mu\nu})Tr(f^{(j)}_{\rho\sigma}\tilde{f}^{(j)}_{\rho\sigma})). \quad (5.7)$$

Here we can apply the low energy approximation such as

$$\begin{aligned}
Tr(f^{(i)}_{\mu\nu}f^{(i)}_{\mu\nu}) &\to \frac{C^4}{l^4_{NC}}\int d^4x tr([D^i_\mu,D^i_\nu][D^i_\mu,D^i_\nu])=\frac{l^4_{NC}}{\pi^2},\\
Tr(f^{(i)}_{\mu\nu}\tilde{f}^{(i)}_{\mu\nu}) &\to \frac{C^4}{2l^4_{NC}}\int d^4x tr\epsilon_{\mu\nu\rho\sigma}([D^i_\mu,D^i_\nu][D^i_\rho,D^i_\sigma])=\pm\frac{l^4_{NC}}{\pi^2},\\
Tr(f^{(i)}_{\mu\nu}f^{(i)}_{\nu\rho}) &\to \frac{C^4}{l^4_{NC}}\int d^4x tr([D^i_\mu,D^i_\nu][D^i_\nu,D^i_\rho])=\frac{l^4_{NC}}{4\pi^2}\delta_{\mu\rho} \qquad (5.8)
\end{aligned}$$

where D^i_μ denotes the covariant derivative of the instanton background which is localized at the i-th block. So the interactions eq.(5.7) can be interpreted due to the exchange of dilaton,

axion and graviton. We have found that the potential between two instantons vanish due to their BPS nature. On the other hand, the following potential is found between an instanton and an anti-instanton

$$-\frac{3}{\pi^4}\frac{l_{NC}^8}{r^8}. \tag{5.9}$$

There is no reason to believe the above approximation when $b < l_{NC}$. In this case the interactions between an instanton and an anti-instanton is well described by the gauge fields which are low energy modes of IIB matrix model. They are close to diagonal degrees of freedom in IIB matrix model. Their contribution can be estimated by gauge theory. The one loop effective potential can be calculated by gauge theory certainly when $b << l_{NC}$

$$\Gamma = -\frac{C^4}{2(4\pi)^2 b^4}\int d^4 x b_8 \tag{5.10}$$

where we have assumed $b\rho >> C$. The above expression is estimated as follows:

$$\begin{aligned}\Gamma &= -\frac{144C^4}{\pi^2 b^4}\int d^4x \frac{\rho^4}{((x-y)^2+\rho^2)^4}\frac{\rho^4}{((x-z)^2+\rho^2)^4} \\ &\sim -\frac{24\rho^4 C^4}{r^8 b^4}\end{aligned} \tag{5.11}$$

where $r = |y - z|$ is assumed to be much larger than ρ. We note that eq.(5.11) falls off with the identical power for large r with eq.(5.9). On the other hand when $b >> l_{NC}$, the standard gauge theory description is no longer valid since we have to take account of the noncommutativity. In fact we have argued that the block-block interaction gives us the correct result. The purpose of this section is to underscore these arguments and explain that the contributions of nonplanar diagrams in noncommutative gauge theory indeed reproduce the block-block interactions. We also provide a more accurate estimate of the crossover scale.

For this purpose, we evaluate the leading term of the effective action by using the plane-wave basis $exp(ik\cdot\hat{x})$ just like the preceding section.

$$\begin{aligned}W &= -\frac{1}{n}\sum_k Tr\left(exp(-ik\cdot\hat{x})\frac{1}{P^2}F_{\mu\nu}\frac{1}{P^2}{}_{\nu\lambda}\frac{1}{P^2}F_{\lambda\rho}\frac{1}{P^2}F_{\rho\mu}exp(ik\cdot\hat{x})\right) \\ &\quad -\frac{2}{n}\sum_k Tr\left(exp(-ik\cdot\hat{x})\frac{1}{P^2}F_{\mu\nu}\frac{1}{P^2}F_{\lambda\rho}\frac{1}{P^2}F_{\mu\rho}\frac{1}{P^2}F_{\lambda\nu}exp(ik\cdot\hat{x})\right) \\ &\quad +\frac{1}{2n}\sum_k Tr\left(exp(-ik\cdot\hat{x})\frac{1}{P^2}F_{\mu\nu}\frac{1}{P^2}F_{\mu\nu}\frac{1}{P^2}F_{\lambda\rho}\frac{1}{P^2}F_{\lambda\rho}exp(ik\cdot\hat{x})\right) \\ &\quad +\frac{1}{4n}\sum_k Tr\left(exp(-ik\cdot\hat{x})\frac{1}{P^2}F_{\mu\nu}\frac{1}{P^2}F_{\lambda\rho}\frac{1}{P^2}F_{\mu\nu}\frac{1}{P^2}F_{\lambda\rho}exp(ik\cdot\hat{x})\right).\end{aligned} \tag{5.12}$$

The above expression is calculated as follows:

$$
\begin{aligned}
W &= n(\frac{1}{2\pi B})^2\int d^4k(\frac{1}{k^2+(Bb)^2})^4 exp(iC^{\mu\nu}k_\mu l_\nu) \\
&\sum_{p,p',q}\frac{3}{2}(-8tr(f^{(i)}_{\mu\nu}(p)f^{(i)}_{\nu\sigma}(p'))tr(f^{(j)}_{\mu\rho}(q)f^{(j)}_{\rho\sigma}(q'))+tr(f^{(i)}_{\mu\nu}(p)f^{(i)}_{\mu\nu}(p'))tr(f^{(j)}_{\rho\sigma}(q)f^{(j)}_{\rho\sigma}(q')) \\
&+tr(f^{(i)}_{\mu\nu}(p)\tilde{f}^{(i)}_{\mu\nu}(p'))tr(f^{(j)}_{\rho\sigma}(q)\tilde{f}^{(j)}_{\rho\sigma}(q'))) \qquad (5.13)
\end{aligned}
$$

where $l=p+p'=q+q'$. Here we have assumed that the external momenta are small compared to the noncommutativity scale. Hence we have dropped the phase which only depends on the external momenta. Note that eq.(5.13) contains the phase factor $exp(iC^{\mu\nu}k_\mu l_\nu)$ due to noncommutativity.

This phase becomes non-trivial when $|k|\sim 1/|l|l_{NC}^2$. We can choose $1/|l|\sim\rho$ in this case. Note that we have another scale $|k|\sim b/l_{NC}^2$ which is associated with the propagator. When $b>\rho>>l_{NC}$, we show that eq.(5.13) can be simply understood in terms of the block-block interactions. It is evaluated as follows:

$$
\begin{aligned}
W &= n(\frac{1}{2\pi B})^2\frac{1}{(Bb)^8}\int d^4k exp(iC^{\mu\nu}k_\mu l_\nu) \\
&\sum_{p,p',q}\frac{3}{2}(-8tr(f^{(i)}_{\mu\nu}(p)f^{(i)}_{\nu\sigma}(p'))tr(f^{(j)}_{\mu\rho}(q)f^{(j)}_{\rho\sigma}(q'))+tr(f^{(i)}_{\mu\nu}(p)f^{(i)}_{\mu\nu}(p'))tr(f^{(j)}_{\rho\sigma}(q)f^{(j)}_{\rho\sigma}(q')) \\
&+tr(f^{(i)}_{\mu\nu}(p)\tilde{f}^{(i)}_{\mu\nu}(p'))tr(f^{(j)}_{\rho\sigma}(q)\tilde{f}^{(j)}_{\rho\sigma}(q'))) \\
&= n^2\frac{1}{(Bb)^8} \\
&\times\sum_{p,q}\frac{3}{2}(-8tr(f^{(i)}_{\mu\nu}(p)f^{(i)}_{\nu\sigma}(-p))tr(f^{(j)}_{\mu\rho}(q)f^{(j)}_{\rho\sigma}(-q))+tr(f^{(i)}_{\mu\nu}(p)f^{(i)}_{\mu\nu}(-p))tr(f^{(j)}_{\rho\sigma}(q)f^{(j)}_{\rho\sigma}(-q)) \\
&+tr(f^{(i)}_{\mu\nu}(p)\tilde{f}^{(i)}_{\mu\nu}(-p))tr(f^{(j)}_{\rho\sigma}(q)\tilde{f}^{(j)}_{\rho\sigma}(-q))) \\
&= \frac{1}{(Bb)^8} \\
&\times\frac{3}{2}(-8(\frac{1}{2\pi C})^2\int d^4x tr(f^{(i)}_{\mu\nu}(x)f^{(i)}_{\nu\sigma}(x))(\frac{1}{2\pi C})^2\int d^4y tr(f^{(j)}_{\mu\rho}(y)f^{(j)}_{\rho\sigma}(y)) \\
&+(\frac{1}{2\pi C})^2\int d^4x tr(f^{(i)}_{\mu\nu}(x)f^{(i)}_{\mu\nu}(x))(\frac{1}{2\pi C})^2\int d^4y tr(f^{(j)}_{\rho\sigma}(y)f^{(j)}_{\rho\sigma}(y)) \\
&+(\frac{1}{2\pi C})^2\int d^4x tr(f^{(i)}_{\mu\nu}(x)\tilde{f}^{(i)}_{\mu\nu}(x))(\frac{1}{2\pi C})^2\int d^4y tr(f^{(j)}_{\rho\sigma}(y)\tilde{f}^{(j)}_{\rho\sigma}(y))). \qquad (5.14)
\end{aligned}
$$

Using the large instanton size approximation eq.(5.8), we indeed reproduce the result of eq.(5.9).

On the other hand, we can neglect the phase factor $exp(iC^{\mu\nu}k_\mu l_\nu)$ when $b<\rho$. With this

approximation, we obtain:

$$
\begin{aligned}
W &= n(\frac{1}{2\pi B})^2 \int d^4k(\frac{1}{k^2+(Bb)^2})^4 \\
&\sum_{p,p',q} \frac{3}{2}(-8tr(f^{(i)}_{\mu\nu}(p)f^{(i)}_{\nu\sigma}(p'))tr(f^{(j)}_{\mu\rho}(q)f^{(j)}_{\rho\sigma}(q')) + tr(f^{(i)}_{\mu\nu}(p)f^{(i)}_{\mu\nu}(p'))tr(f^{(j)}_{\rho\sigma}(q)f^{(j)}_{\rho\sigma}(q')) \\
&+tr(f^{(i)}_{\mu\nu}(p)\tilde{f}^{(i)}_{\mu\nu}(p'))tr(f^{(j)}_{\rho\sigma}(q)\tilde{f}^{(j)}_{\rho\sigma}(q'))) \\
&= -\frac{C^4}{2(4\pi)^2 b^4}\int d^4x b_8.
\end{aligned}
\tag{5.15}
$$

In this way, we reproduce the ordinary gauge theory result eq.(5.11).

6 Conclusions

In this paper we have proposed a bi-local field representation of noncommutative field theories. In the momentum eigenstate representation, the momenta of fields can become much larger than the noncommutative scale l_{NC}. However we can no longer regard these large momentum degrees of freedom as ordinary local fields since the wave functions become noncommutative. Due to the noncommutativity of space-time, the wave functions with large momenta are more naturally interpreted as translation operators and hence those degrees of freedom are interpreted as bi-local fields. In the $n \times n$ matrix reguralization of noncommutative field theories, there are n^2 degrees of freedom and this number is much larger than n which is the degrees of freedom of ordinary local field with UV cutoff l_{NC}^{-1}. This implies that noncommutative field theories contain infinitely many particles. It is interesting to investigate the possibility that these infinite degrees of freedom may pile up to strings. [5]

We have also calculated long range interactions in noncommutative field theories. In the conventional approach, they manifest as IR singular behaviors of nonplanar diagrams. We have seen that this feature can be naturally understood in terms of block-block interactions in the matrix model picture. This type of interactions are characteristic to matrix models, not restricted to twisted reduced models which are directly related to noncommutative field theories. In the case of type IIB matrix model, we have shown [1] the appearance of block-block interactions in generic backgrounds and interpreted them as propagations of massless supergravity multiplets. These interactions differ from those of the ordinary gauge theory such as $D=4$ $\mathcal{N}=4$ super Yang-Mills field theory as is explained in the preceding section. This point demonstrate the richness of matrix models over the corresponding field theories

[5] see also [25]

and it is one of advantages to consider IIB matrix model as a constructive formulation of superstring.

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